

The square lattice Ising model on the rectangle

I: Finite systems

Alfred Hucht

Faculty for Physics, University of Duisburg-Essen, 47058 Duisburg, Germany

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Abstract

The partition function of the square lattice Ising model on the rectangle is calculated exactly for arbitrary system size $L \times M$ and temperature. We start with the dimer method of Kasteleyn, McCoy & Wu, construct a highly symmetric block transfer matrix and derive a factorization of the involved determinant, effectively decomposing the free energy of the system into two parts, $F(L, M) = F_{\text{strip}}(L, M) + F_{\text{strip}}^{\text{res}}(L, M)$, where the residual part $F_{\text{strip}}^{\text{res}}(L, M)$ contains the nontrivial finite- L contributions for fixed M . It is given by the determinant of a $M/2 \times M/2$ matrix and can be mapped onto an effective spin model with M Ising spins and long-range interactions. While $F_{\text{strip}}^{\text{res}}(L, M)$ becomes exponentially small for large L/M or off-critical temperatures, it leads to important finite-size effects such as the critical Casimir force near criticality. The relations to the Casimir potential and the Casimir force are discussed.

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I. INTRODUCTION

The two-dimensional Ising model [1] on the $L \times M$ square lattice is one of the best investigated models in statistical mechanics. After the exact solution of the periodic case by Onsager [2], many authors have contributed to the knowledge about this model under various aspects, such as different boundary conditions (BCs) or surface effects [3, 4]. Near the critical temperature T_c , where the correlation length $\xi(T)$ of thermal fluctuations becomes of the order of the system size L or M in finite systems, interesting finite-size effects such as the critical Casimir effect emerge, which describes an interaction of the system boundaries mediated by long-range critical fluctuations [5] in close analogy to the quantum electrodynamical Casimir effect [6]. These finite-size effects can be described by universal finite-size scaling functions, that only depend on the bulk and surface universality classes of the model, as well as on the BCs and on the system shape. They have been calculated exactly for many cases, albeit mostly in strip geometry, where the aspect ratio $\rho = L/M$ of the system goes to zero [7–9]. Directly at the critical point, exact methods or conformal field theory can be used to get exact expressions for the Casimir amplitude $\Delta_C(\rho)$ for arbitrary ρ . This has been done for periodic [10, 11] as well as for open BCs [12]. At arbitrary aspect ratios and temperatures, however, the finite-size scaling functions must be derived from the exact solution of the system with the correct BC. For the Ising model, this has been done only in a few cases, namely for the torus with periodic BCs in both directions [13, 14] and for the cylinder with open BCs in one direction [14].

In this work and in the forthcoming publication [15] we will present a calculation of these finite-size contributions, namely the residual free energy also denoted Casimir potential, as well as the resulting critical Casimir forces, for open BCs at arbitrary temperatures T and system size $L \times M$. In order to calculate these quantities correctly, all infinite volume free energies, i. e., the bulk free energy $LMf_b(T)$, the surface free energies $Lf_s^{\leftrightarrow}(T)$ and $Mf_s^{\updownarrow}(T)$ in the two directions \leftrightarrow and \updownarrow , as well as the corner free energy $f_c(T)$ must be known and subtracted from the free energy of the finite system. While the bulk and surface free energies are known for a long time [2, 3], the corner free energy $f_c(T)$ was only known below T_c from a conjecture by Vernier & Jacobsen [16]. The corresponding product formula for the paramagnetic phase is given in the Appendix of this work and will be discussed in [15].

In a recent preprint [17], Rodney J. Baxter presented an exact calculation of the infinite

volume corner free energy $f_c(T)$ in the ordered phase $T < T_c$, verifying the conjecture of Vernier & Jacobsen. In this manuscript we present a calculation within the same model and geometry and discuss the similarities and differences. While Baxter focused on the corner free energy contribution $f_c(T)$ in the thermodynamic limit, the focus of this work is on the exact finite-size corrections to the free energy at arbitrary system size and temperature.

We start the present calculation with the Pfaffian formulation of Kasteleyn, McCoy & Wu [3, 18] in cylinder geometry and reduce the involved determinant of a sparse $4LM \times 4LM$ matrix to the determinant of a $LM \times LM$ block-tridiagonal matrix using an appropriate Schur complement. This determinant can then be calculated with the formula of Molinari [19], introducing 2×2 block transfer matrices \mathcal{T}_ℓ with $M \times M$ blocks. Up to here the calculation is done for arbitrary local couplings $K_{\ell,m}^{\leftrightarrow}$ and $K_{\ell,m}^{\updownarrow}$ in the two directions on the cylinder. Then we assume open BCs in both directions and homogeneous, albeit anisotropic couplings K^{\leftrightarrow} and K^{\updownarrow} . After that simplification the partition function Z is of the form $Z^2 \propto \det(\mathbf{1} \mathbf{0} | \mathcal{T}^L | \mathbf{1} \mathbf{0})$, in strong analogy to Baxter's result [17].

While Baxter at this point performs the thermodynamic limit $L \rightarrow \infty$ with constant M , neglecting the finite- L contributions, we are able to proceed and further reduce the size of the involved matrices. The block transfer matrix \mathcal{T} can be symmetrized and block diagonalized such that its eigenvalues λ are real and occur in pairs (λ, λ^{-1}) , and the calculation is simplified by the introduction of the natural angle variable φ , leading to the characteristic polynomial $P_M(\varphi)$. It turns out that the eigenvalues λ are directly related to the well-known Onsager- γ via $\gamma = \log \lambda$.

The eigenvectors \vec{X} of \mathcal{T} show an important symmetry with respect to the mapping $\lambda \leftrightarrow \lambda^{-1}$, which can eventually be used to reduce the involved matrices from $2M \times 2M$ to $M \times M$ and, more important, to factorize the determinant into a product of the form $\det(\mathbf{W}^T \mathbf{D} \mathbf{W}) = \det^2 \mathbf{W} \det \mathbf{D}$, where \mathbf{D} is diagonal.

The remaining matrix \mathbf{W} is of Vandermonde type and can be considerably simplified using the invariance property of Vandermonde determinants with respect to basis transformations. Using the well known product formula for these determinants the matrix size can be further reduced to $M/2 \times M/2$. We show that this determinant contains all remaining nontrivial finite-size contributions, and discuss the different resulting contributions to the free energy.

Finally we present an exact mapping of the remaining determinant onto a long-range spin model with M spins and logarithmic interactions in an effective magnetic field of strength

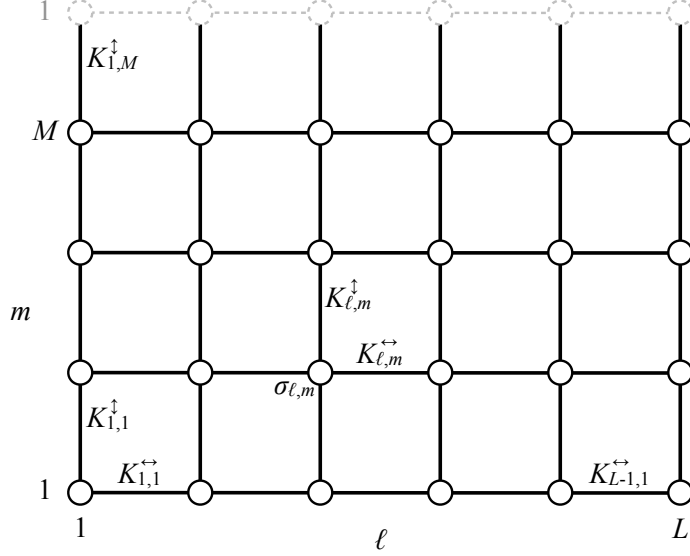


Figure 1. The square lattice with cylinder geometry for $M = 4$ and $L = 6$.

L , which might give rise to an alternative calculation of the remaining determinant. We conclude with a discussion of the results.

In the second part of this work [15], which will be published separately, we perform the finite-size scaling limit $L, M \rightarrow \infty$, $T \rightarrow T_c$ with fixed temperature scaling variables $x \propto (T/T_c - 1)M$, and aspect ratio ρ . After a number of simplifications, we derive an exponentially fast converging series for the Casimir scaling functions. At the critical point $T = T_c$ we can rewrite the Casimir amplitude $\Delta_C(\rho)$ in terms of the Dedekind eta function, confirming a prediction from conformal field theory [12].

II. MODEL AND PFAFFIAN REPRESENTATION

We consider an Ising model on the square lattice with L columns and M rows as shown in Fig. 1, and start with arbitrary reduced (in units of $k_B T$, with Boltzmann constant k_B) couplings $K_{\ell,m}^{\leftrightarrow}$ and $K_{\ell,m}^{\uparrow}$ in horizontal and vertical direction on the cylinder periodic in vertical (M) direction. Our aim is to calculate the partition function

$$Z = \text{Tr} \exp \sum_{\ell=1}^L \sum_{m=1}^M \left(K_{\ell,m}^{\leftrightarrow} \sigma_{\ell,m} \sigma_{\ell+1,m} + K_{\ell,m}^{\uparrow} \sigma_{\ell,m} \sigma_{\ell,m+1} \right), \quad (1)$$

where the trace is over all 2^{LM} configurations of the LM spins $\sigma_{\ell,m}$, with $\sigma_{L+1,m} = \sigma_{1,m}$ and $\sigma_{\ell,M+1} = \sigma_{\ell,1}$. We assume open BC in horizontal (L) direction, $K_{L,m}^{\leftrightarrow} = 0$, and first derive

a transfer matrix formulation for this general case. After that we focus on the rectangular homogeneous case, $K_{\ell,M}^{\uparrow} = 0$, $K_{\ell,m < M}^{\uparrow} = K^{\uparrow}$, $K_{\ell < L,m}^{\leftrightarrow} = K^{\leftrightarrow}$, where we still allow for anisotropic couplings.

Our starting point is the Pfaffian representation by Kasteleyn, McCoy & Wu [3, 18], where the partition function in cylinder geometry is given by

$$Z = \sqrt{C_0} \text{Pf } \mathcal{A} = \sqrt{C_0 \det \mathcal{A}}, \quad (2)$$

with the constant

$$C_0 = 4^{LM} \prod_{\ell=1}^{L-1} \prod_{m=1}^M \cosh^2 K_{\ell,m}^{\leftrightarrow} \prod_{\ell=1}^L \prod_{m=1}^M \cosh^2 K_{\ell,m}^{\uparrow}. \quad (3)$$

We define the antisymmetric $4LM \times 4LM$ sparse matrix \mathcal{A} as a 4×4 block matrix (the bar denotes transposition, “ \equiv ” denotes a definition)

$$\mathcal{A} \equiv \begin{bmatrix} \mathbf{0} & \mathbf{1} + \mathbf{Z}^{\uparrow} & -\mathbf{1} & -\mathbf{1} \\ -\mathbf{1} - \bar{\mathbf{Z}}^{\uparrow} & \mathbf{0} & \mathbf{1} & -\mathbf{1} \\ \mathbf{1} & -\mathbf{1} & \mathbf{0} & \mathbf{1} + \mathbf{Z}^{\leftrightarrow} \\ \mathbf{1} & \mathbf{1} & -\mathbf{1} - \bar{\mathbf{Z}}^{\leftrightarrow} & \mathbf{0} \end{bmatrix}, \quad (4)$$

where the $LM \times LM$ matrices \mathbf{Z}^{δ} contain the couplings $z_{\ell,m}^{\delta} \equiv \tanh K_{\ell,m}^{\delta}$ in direction $\delta = \leftrightarrow, \uparrow$ via the $M \times M$ and $LM \times LM$ diagonal matrices

$$\mathbf{z}_{\ell}^{\delta} \equiv \text{diag}(z_{\ell,1}^{\delta}, \dots, z_{\ell,M}^{\delta}), \quad \mathbf{Z}^{\delta} \equiv \text{diag}(\mathbf{z}_1^{\delta}, \dots, \mathbf{z}_L^{\delta}), \quad (5)$$

according to

$$\mathbf{Z}^{\leftrightarrow} = \mathbf{z}^{\leftrightarrow}(\mathbf{H}_L^0 \otimes \mathbf{1}_M), \quad (6a)$$

$$\mathbf{Z}^{\uparrow} = \mathbf{z}^{\uparrow}(\mathbf{1}_L \otimes \mathbf{H}_M^-) = \text{diag}(\mathbf{z}_1^{\uparrow} \mathbf{H}_M^-, \dots, \mathbf{z}_L^{\uparrow} \mathbf{H}_M^-). \quad (6b)$$

Here we have introduced the $n \times n$ shift matrices

$$\mathbf{H}_n^0 \equiv \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}, \quad \mathbf{H}_n^- \equiv \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -1 & & & 0 \end{pmatrix}, \quad (7)$$

that, together with the $n \times n$ identity matrix $\mathbf{1}_n$, define the lattice structure. We drop the index n from unit and zero matrices $\mathbf{1}$, $\mathbf{0}$ as long as it can be implied from the context.

III. SCHUR REDUCTION

We first reduce the matrix size from $4LM \times 4LM$ to $LM \times LM$ by a standard Schur reduction according to

$$\det \mathcal{A} = \det \mathcal{A}_{\bar{i}, \bar{i}} \det \mathcal{C}_{i, i}, \quad (8)$$

where \bar{i} denotes the index complement of i , i.e., $\mathcal{A}_{\bar{i}, j}$ is derived from \mathcal{A} by dropping row i and taking column j . We choose $i = 4$ to find, for even M ,

$$\det \mathcal{A}_{\bar{4}, \bar{4}} = \prod_{\ell=1}^L \left(\prod_{\substack{m=1 \\ m \text{ odd}}}^{M-1} z_{\ell, m}^{\uparrow\downarrow} + \prod_{\substack{m=2 \\ m \text{ even}}}^M z_{\ell, m}^{\uparrow\downarrow} \right)^2 \quad (9)$$

as well as the $LM \times LM$ Schur complement

$$\mathcal{C}_{4,4} \equiv \mathcal{A} / \mathcal{A}_{\bar{4}, \bar{4}} \equiv \mathcal{A}_{4,4} - \mathcal{A}_{4, \bar{4}} \mathcal{A}_{\bar{4}, \bar{4}}^{-1} \mathcal{A}_{\bar{4}, 4}, \quad (10)$$

which is antisymmetric and block tridiagonal,

$$\mathcal{C}_{4,4} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 & & \\ -\bar{\mathbf{B}}_1 & \ddots & \ddots & \\ & \ddots & \ddots & \mathbf{B}_{L-1} \\ & & -\bar{\mathbf{B}}_{L-1} & \mathbf{A}_L \end{bmatrix}, \quad (11)$$

with $M \times M$ matrices \mathbf{A}_ℓ and \mathbf{B}_ℓ . We also could have chosen $i = 3$ for the reduction, which would reflect the matrix $\mathcal{C}_{i,i}$ along the anti-diagonal, whereas the indices $i = 1, 2$ do not lead to block tridiagonal matrices $\mathcal{C}_{i,i}$. The explicit expressions for the matrices \mathbf{A}_ℓ and \mathbf{B}_ℓ are

$$\mathbf{B}_\ell^{-1} = -(\mathbf{z}_\ell^{\leftrightarrow})^{-1} \mathbf{D}_\ell, \quad (12a)$$

$$\mathbf{A}_1 = \mathbf{A}_1^-, \quad (12b)$$

$$\mathbf{A}_{\ell>1} = \mathbf{A}_\ell^- + \mathbf{z}_{\ell-1}^{\leftrightarrow} \mathbf{A}_{\ell-1}^+ \mathbf{z}_{\ell-1}^{\leftrightarrow}, \quad (12c)$$

with the auxiliary matrices

$$\mathbf{A}_\ell^\pm \equiv \pm \left[(\mathbf{1} \pm \bar{\mathbf{Z}}_\ell^\uparrow)^{-1} - (\mathbf{1} \pm \mathbf{Z}_\ell^\uparrow)^{-1} \right]^{-1}, \quad (13a)$$

$$\mathbf{D}_\ell \equiv (\mathbf{1} - \bar{\mathbf{Z}}_\ell^\uparrow)(\mathbf{1} - \mathbf{Z}_\ell^\uparrow \bar{\mathbf{Z}}_\ell^\uparrow)^{-1} - (\mathbf{1} - \mathbf{Z}_\ell^\uparrow)(\mathbf{1} - \bar{\mathbf{Z}}_\ell^\uparrow \mathbf{Z}_\ell^\uparrow)^{-1}, \quad (13b)$$

where $\mathbf{Z}_\ell^\uparrow = \mathbf{z}_\ell^\uparrow \mathbf{H}_M^-$ from Eq. (6b). As the matrices \mathbf{B}_ℓ are invertible, the remaining determinant $\det \mathcal{C}_{4,4}$ can be calculated with a transfer matrix approach.

IV. THE BLOCK TRANSFER MATRIX \mathcal{T}

The determinant of the block tridiagonal matrix $\mathcal{C}_{4,4}$ from Eq. (11) can be calculated with the method of Molinari [19]. We introduce the 2×2 block transfer matrix (TM)

$$\mathcal{T}_{\ell,\ell-1}^\dagger \equiv \begin{bmatrix} -\mathbf{B}_\ell^{-1}\mathbf{A}_\ell & \mathbf{B}_\ell^{-1}\bar{\mathbf{B}}_{\ell-1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}, \quad (14)$$

$\ell = 1, \dots, L$, and formally define \mathbf{B}_0 and \mathbf{B}_L , with $\mathbf{z}_0^\delta = \mathbf{z}_L^\dagger = \mathbf{0}$ and $\mathbf{z}_L^{\leftrightarrow} = \mathbf{1}$, in order to keep the expressions simple. We can factorize $\mathcal{T}_{\ell,\ell-1}^\dagger$ into two parts depending on ℓ and $\ell-1$, respectively,

$$\mathcal{T}_{\ell,\ell-1}^\dagger = \begin{bmatrix} (\mathbf{z}_\ell^{\leftrightarrow})^{-1}\mathbf{D}_\ell\mathbf{A}_\ell^- & (\mathbf{z}_\ell^{\leftrightarrow})^{-1}\mathbf{D}_\ell \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{z}_{\ell-1}^{\leftrightarrow}\mathbf{A}_{\ell-1}^+ \mathbf{z}_{\ell-1}^{\leftrightarrow} & \mathbf{z}_{\ell-1}^{\leftrightarrow}\bar{\mathbf{D}}_{\ell-1}^{-1} \end{bmatrix} \equiv \mathcal{T}_\ell^{(1)}\mathcal{T}_{\ell-1}^{(2)}, \quad (15)$$

and we observe that in the product of TMs, $\dots \mathcal{T}_{\ell+1,\ell}^\dagger \mathcal{T}_{\ell,\ell-1}^\dagger \dots = \dots \mathcal{T}_{\ell+1}^{(1)} \mathcal{T}_\ell^{(2)} \mathcal{T}_\ell^{(1)} \mathcal{T}_{\ell-1}^{(2)} \dots$, we can identify a shifted TM $\mathcal{T}_\ell^\dagger \equiv \mathcal{T}_\ell^{(2)}\mathcal{T}_\ell^{(1)}$, depending only on ℓ , with the factorization

$$\mathcal{T}_\ell^\dagger \equiv \mathcal{T}_\ell^{(2)}\mathcal{T}_\ell^{(1)} = \begin{bmatrix} (\mathbf{z}_\ell^{\leftrightarrow})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{z}_\ell^{\leftrightarrow} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{A}_\ell^+ & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{D}_\ell \\ \bar{\mathbf{D}}_\ell^{-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{A}_\ell^- & \mathbf{1} \end{bmatrix}. \quad (16)$$

Using a block rotation by $\theta = \pi/4$, with

$$\mathbf{R}_\theta \equiv \mathbf{r}_\theta \otimes \mathbf{1}, \quad \mathbf{r}_\theta \equiv \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad (17)$$

we find the simple representation

$$\bar{\mathbf{R}}_{\frac{\pi}{4}} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{A}_\ell^+ & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{D}_\ell \\ \bar{\mathbf{D}}_\ell^{-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{A}_\ell^- & \mathbf{1} \end{bmatrix} \mathbf{R}_{\frac{\pi}{4}} = \begin{bmatrix} \bar{\mathbf{H}}^- & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{t}_{\ell,+} & \mathbf{t}_{\ell,-} \\ \mathbf{t}_{\ell,-} & \mathbf{t}_{\ell,+} \end{bmatrix} \begin{bmatrix} \mathbf{H}^- & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \equiv \mathcal{V}_\ell^\dagger, \quad (18)$$

where the matrices

$$\mathbf{t}_\ell \equiv \text{diag}(t_{\ell,1}, \dots, t_{\ell,M}) \quad (19)$$

contain the dual couplings $t \equiv z^{\dagger*} = \frac{1-z^\dagger}{1+z^\dagger}$ of z^\dagger , and we have introduced the abbreviation

$$a_\pm \equiv \frac{1}{2}(a \pm a^{-1}), \quad (20)$$

such that $a^{\pm 1} = a_+ \pm a_-$, for couplings and other quantities. From here on we express the vertical couplings z^\dagger through their dual couplings t , and simply write z for the horizontal couplings z^{\leftrightarrow} . Note that our z is denoted u in [17].

Inserting three $\mathbf{1}$ s into Eq. (16), we find

$$\begin{aligned}\mathcal{T}_\ell^\dagger &= \mathbf{R}_{\frac{\pi}{4}} \bar{\mathbf{R}}_{\frac{\pi}{4}} \underbrace{\begin{bmatrix} \mathbf{z}_\ell^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{z}_\ell \end{bmatrix}}_{\mathbf{v}_\ell^{\leftrightarrow}} \underbrace{\mathbf{R}_{\frac{\pi}{4}} \bar{\mathbf{R}}_{\frac{\pi}{4}} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{A}_\ell^+ & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{D}_\ell \\ \bar{\mathbf{D}}_\ell^{-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{A}_\ell^- & \mathbf{1} \end{bmatrix}}_{\mathbf{v}_\ell^\dagger} \mathbf{R}_{\frac{\pi}{4}} \bar{\mathbf{R}}_{\frac{\pi}{4}} \\ &= \mathbf{R}_{\frac{\pi}{4}} \mathbf{v}_\ell^{\leftrightarrow} \mathbf{v}_\ell^\dagger \bar{\mathbf{R}}_{\frac{\pi}{4}},\end{aligned}\tag{21}$$

with

$$\mathbf{v}_\ell^{\leftrightarrow} = \begin{bmatrix} \mathbf{z}_{\ell,+} & -\mathbf{z}_{\ell,-} \\ -\mathbf{z}_{\ell,-} & \mathbf{z}_{\ell,+} \end{bmatrix}\tag{22}$$

in analogy to Eq. (18). Following [19], the determinant Eq. (8) becomes

$$\begin{aligned}\det \mathcal{A} &= C_1 \det \langle \mathbf{1} \mathbf{0} | \mathcal{T}_{L,L-1}^\dagger \mathcal{T}_{L-1,L-2}^\dagger \cdots \mathcal{T}_{2,1}^\dagger \mathcal{T}_{1,0}^\dagger | \mathbf{1} \mathbf{0} \rangle \\ &= C_1 \det \langle \mathbf{1} \mathbf{0} | \mathcal{T}_L^\dagger \mathcal{T}_{L-1}^\dagger \cdots \mathcal{T}_2^\dagger \mathcal{T}_1^\dagger | \mathbf{1} \mathbf{0} \rangle \\ &= C_1 \det \langle \mathbf{e} | \mathbf{v}_L^{\leftrightarrow} \mathbf{v}_L^\dagger \mathbf{v}_{L-1}^{\leftrightarrow} \mathbf{v}_{L-1}^\dagger \cdots \mathbf{v}_2^{\leftrightarrow} \mathbf{v}_2^\dagger \mathbf{v}_1^{\leftrightarrow} \mathbf{v}_1^\dagger | \mathbf{e} \rangle,\end{aligned}\tag{23}$$

with $|\mathbf{e}\rangle \equiv \bar{\mathbf{R}}_{\frac{\pi}{4}} |\mathbf{1} \mathbf{0}\rangle = \frac{1}{\sqrt{2}} |\mathbf{1} \mathbf{1}\rangle$ and constant

$$C_1 \equiv \det \mathcal{A}_{\bar{4},\bar{4}} \prod_{\ell=1}^L \det \mathbf{B}_\ell = \prod_{\ell=1}^{L-1} \prod_{m=1}^M z_{\ell,m}^{\leftrightarrow} \prod_{\ell=1}^L \prod_{m=1}^M (1 - z_{\ell,m}^{\dagger 2}).\tag{24}$$

Here and in the following we use bra-ket notation for the boundary block vectors, such that $\langle \mathbf{e} |$ and $|\mathbf{e}\rangle$ are $M \times 2M$ and $2M \times M$ dimensional matrices, respectively, and $\langle \mathbf{1} \mathbf{0} | \mathcal{T} | \mathbf{1} \mathbf{0} \rangle$ gives the 1,1-element of block matrix \mathcal{T} .

The final result for the partition function, Eq. (2), with arbitrary couplings reads

$$Z = \sqrt{C_2^\dagger Z^\dagger},\tag{25a}$$

with

$$Z^\dagger \equiv \det \langle \mathbf{e} | \mathbf{v}_L^\dagger \mathbf{v}_{L-1}^{\leftrightarrow} \mathbf{v}_{L-1}^\dagger \cdots \mathbf{v}_2^{\leftrightarrow} \mathbf{v}_2^\dagger \mathbf{v}_1^{\leftrightarrow} \mathbf{v}_1^\dagger | \mathbf{e} \rangle,\tag{25b}$$

as $\mathbf{v}_L^{\leftrightarrow} = \mathbf{1}$, and with constant

$$C_2^\dagger \equiv C_0 C_1 = 2^{(L+1)M} \prod_{\ell=1}^{L-1} \prod_{m=1}^M \frac{1}{z_{\ell,m,-}}.\tag{25c}$$

This result is valid for arbitrary couplings on the cylinder, and it is straightforward to derive an analog expression for the torus. We point out that we can “transpose” both $\mathbf{v}_\ell^{\leftrightarrow}$ and \mathbf{v}_ℓ^\dagger

from 2×2 block structure with $M \times M$ blocks to $M \times M$ block structure with 2×2 blocks to get, for $M = 4$,

$$\hat{\mathbf{V}}_\ell^{\leftrightarrow} = \begin{pmatrix} z_{\ell,1,+} & -z_{\ell,1,-} & & & & & & \\ -z_{\ell,1,-} & z_{\ell,1,+} & & & & & & \\ & & z_{\ell,2,+} & -z_{\ell,2,-} & & & & \\ & & -z_{\ell,2,-} & z_{\ell,2,+} & & & & \\ & & & & z_{\ell,3,+} & -z_{\ell,3,-} & & \\ & & & & -z_{\ell,3,-} & z_{\ell,3,+} & & \\ & & & & & & z_{\ell,4,+} & -z_{\ell,4,-} \\ & & & & & & -z_{\ell,4,-} & z_{\ell,4,+} \end{pmatrix}, \quad (26a)$$

$$\hat{\mathbf{V}}_\ell^{\updownarrow} = \begin{pmatrix} t_{\ell,4,+} & & & & & & & -t_{\ell,4,-} \\ & t_{\ell,1,+} & t_{\ell,1,-} & & & & & \\ & t_{\ell,1,-} & t_{\ell,1,+} & & & & & \\ & & & t_{\ell,2,+} & t_{\ell,2,-} & & & \\ & & & t_{\ell,2,-} & t_{\ell,2,+} & & & \\ & & & & & t_{\ell,3,+} & t_{\ell,3,-} & \\ & & & & & t_{\ell,3,-} & t_{\ell,3,+} & \\ -t_{\ell,4,-} & & & & & & & t_{\ell,4,+} \end{pmatrix}. \quad (26b)$$

We observe the intuitive picture that alternating applications $|\hat{\Psi}\rangle \leftarrow \hat{\mathbf{V}}_\ell^{\updownarrow}|\hat{\Psi}\rangle$ and $|\hat{\Psi}\rangle \leftarrow \hat{\mathbf{V}}_\ell^{\leftrightarrow}|\hat{\Psi}\rangle$ on the state vector $|\hat{\Psi}\rangle$ lead to a repetitive mixing of its components $|\hat{\Psi}\rangle_m$ with left and right neighbor entries $|\hat{\Psi}\rangle_{m\pm 1}$. We now focus on the case of open BCs in both directions and homogeneous anisotropic couplings.

V. OPEN BOUNDARY CONDITIONS AND SYMMETRY

For homogeneous anisotropic couplings $z_{\ell < L, m} = z$, $z_{L, m} = 1$, $t_{\ell, m < M} = t$ and open BCs $t_{\ell, M} = 1$ also in vertical direction we define the symmetric 2×2 block transfer matrix

$$\mathcal{T}_2 \equiv \begin{bmatrix} \mathcal{T}_+ & \mathcal{T}_- \\ \mathcal{T}_- & \mathcal{T}_+ \end{bmatrix} \equiv \mathbf{S}_2 \mathbf{V}_{\leftrightarrow}^{1/2} \mathbf{V}_{\updownarrow} \mathbf{V}_{\leftrightarrow}^{1/2} \mathbf{S}_2, \quad (27)$$

where we employed a unitary reversal of the second row and column with

$$\mathbf{S}_2 \equiv \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{bmatrix}, \quad \mathbf{S} \equiv \begin{pmatrix} & & & 1 \\ & & & \\ & & \ddots & \\ 1 & & & \end{pmatrix}, \quad (28)$$

in order to achieve the highly symmetric structure of \mathcal{T}_2 . Below it will become clear why we denote the two different blocks \mathcal{T}_\pm . In terms of \mathcal{T}_2 the partition function Eq. (25b) becomes

$$\mathcal{Z} \equiv z^{-M} \mathcal{Z}^\dagger = \det\langle \mathbf{e}_2 | \mathcal{T}_2^L | \mathbf{e}_2 \rangle, \quad (29a)$$

with modified boundary state

$$|\mathbf{e}_2\rangle \equiv \frac{1}{\sqrt{z}} \mathbf{S}_2 \mathbf{V}_{\leftrightarrow}^{-1/2} |\mathbf{e}\rangle = \frac{1}{\sqrt{2}} |\mathbf{1} \mathbf{S}\rangle. \quad (29b)$$

Note that we have moved an extra factor z^M into $C_2 \equiv z^M C_2^\dagger$ to get $|\mathbf{e}_2\rangle$ independent of z .

The two symmetric $M \times M$ blocks of \mathcal{T}_2 are

$$\mathcal{T}_+ = \begin{pmatrix} a_0^+ & c & & & \\ c & a & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & a & c \\ & & & c & a_0^- \end{pmatrix}, \quad \mathcal{T}_- = \begin{pmatrix} & & & d^- & b_0 \\ & & \ddots & b & d^+ \\ & \ddots & \ddots & \ddots & \\ d^- & b & \ddots & & \\ b_0 & d^+ & & & \end{pmatrix}, \quad (30a)$$

with matrix elements (c.f. Eq. (20))

$$\begin{aligned} a &= t_+ z_+ & b &= -t_+ z_- \\ a_0^\pm &= t_+ z_+ + \frac{1}{2}(1 - t_+)(z_+ \pm 1) & b_0 &= -\frac{1}{2}(1 + t_+)z_- \\ c &= -\frac{1}{2}t_- z_- & d^\pm &= \pm \frac{1}{2}t_- (1 \pm z_+). \end{aligned} \quad (30b)$$

Note that a matrix like \mathcal{T}_2 , with X-shaped structure, is sometimes called a “cruciform matrix” and also occurs in the dimer problem with open BCs [20]. However, here the components are tridiagonal and slightly more complicated.

We now turn to the eigensystem $\mathcal{T}_2 \vec{X}_\lambda = \lambda \vec{X}_\lambda$ of \mathcal{T}_2 . Due to the inversion symmetry

$$\mathcal{T}_2^{-1} = \begin{bmatrix} \mathcal{T}_+ & -\mathcal{T}_- \\ -\mathcal{T}_- & \mathcal{T}_+ \end{bmatrix} \quad (31)$$

the $2M$ eigenvalues λ occur in pairs λ, λ^{-1} , and the unitary matrix of normalized eigenvectors $(\mathbf{X})_{\lambda,m} \equiv (\vec{X}_\lambda)_m$ can be written as the direct product

$$\mathbf{X} = \mathbf{r}_{\frac{\pi}{4}} \otimes \mathbf{x}, \quad (32)$$

with rotation matrix \mathbf{r}_θ from Eq. (17), provided that we sort the eigenvalues λ of \mathcal{T}_2 in proper order $\{\lambda_1, \dots, \lambda_M, \lambda_1^{-1}, \dots, \lambda_M^{-1}\}$, see below for details on the ordering. Using the $M \times M$ matrix \mathbf{x} together with the corresponding diagonal matrix of eigenvalues,

$$\mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \dots, \lambda_M), \quad (33)$$

we can define a $M \times M$ transfer matrix

$$\mathcal{T} \equiv \bar{\mathbf{x}} \mathbf{\Lambda} \mathbf{x} \quad (34)$$

such that Eqs. (27) and (32) give

$$\mathcal{T}_{\pm} = \frac{1}{2} (\mathcal{T} \pm \mathcal{T}^{-1}) \quad \Leftrightarrow \quad \mathcal{T}^{\pm 1} = \mathcal{T}_+ \pm \mathcal{T}_-. \quad (35)$$

Remarkably, we find $\det \mathbf{\Lambda} = \det \mathcal{T} = t$. Note that the \pm notation is as defined in Eq. (20).

We can interpret the steps above as a block diagonalization of \mathcal{T}_2 through a rotation with \mathbf{R}_{θ} from Eq. (17) according to

$$\mathbf{R}_{\frac{\pi}{4}} \mathcal{T}_2 \bar{\mathbf{R}}_{\frac{\pi}{4}} = \begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T}^{-1} \end{bmatrix}. \quad (36)$$

Nonetheless, we first proceed with the simpler tridiagonal matrix \mathcal{T}_+ from Eq. (30a). The eigenvalues of \mathcal{T}_{\pm} fulfill $\mathcal{T}_{\pm} = \lambda_{\pm} \mathbf{x}$, and we can analyze the eigensystem of \mathcal{T}_+ instead of \mathcal{T}_2 or \mathcal{T} , which is much easier. The eigenvalues λ and λ_{\pm} are directly related to the Onsager- γ via

$$\lambda = e^{\gamma}, \quad \lambda_+ = \cosh \gamma, \quad \lambda_- = \sinh \gamma. \quad (37)$$

VI. EIGENVALUES OF \mathcal{T} AND THE ANGLE φ

The characteristic polynomial of the matrix \mathcal{T}_+ ,

$$P_M(\lambda_+) \equiv \det(\mathcal{T}_+ - \lambda_+ \mathbf{1}), \quad (38)$$

is derived from Eqs. (30) using the well known recursion formula for tridiagonal matrices (see, e.g., [19]),

$$P_M(\lambda_+) = \langle a_0^- - \lambda_+, c | \begin{pmatrix} a - \lambda_+ & c \\ -c & 0 \end{pmatrix}^{M-2} | a_0^+ - \lambda_+, -c \rangle \quad (39)$$

$$= \left(\frac{t_- z_-}{2} \right)^M \langle 1, -t^* z^* | \mathbf{Q}^M | 1, t^*/z^* \rangle, \quad (40)$$

with

$$\mathbf{Q} = \begin{pmatrix} 2 \frac{t_+ z_+ - \lambda_+}{t_- z_-} & -1 \\ 1 & 0 \end{pmatrix}. \quad (41)$$

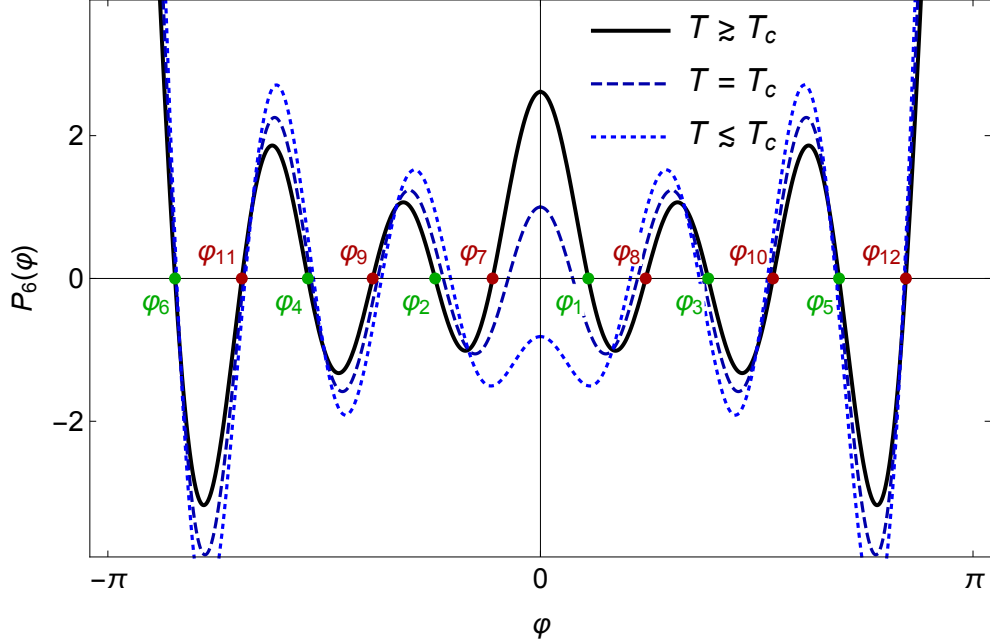


Figure 2. Characteristic polynomial $P_M(\varphi)$, Eq. (45), for $M = 6$ above, at, and below T_c . The eigenvalues are ordered as shown (see text).

The eigenvalues of \mathbf{Q} ,

$$q^\pm = \frac{t_+ z_+ - \lambda_+}{t_- z_-} \mp \frac{\sqrt{(t_+ z_+ - \lambda_+)^2 - t_-^2 z_-^2}}{t_- z_-} \quad (42)$$

have modulus one and can be written as $q^\pm = e^{\pm i\varphi}$, if we define the angle φ such that

$$\cos \varphi = \frac{t_+ z_+ - \lambda_+}{t_- z_-}, \quad \sin \varphi = i \frac{\sqrt{t z - \lambda} \sqrt{1 - t z \lambda} \sqrt{z - t \lambda} \sqrt{t - z \lambda}}{2 t z \lambda t_- z_-}. \quad (43)$$

Note that the factorization of the square root determines the sign of $\sin \varphi$. Then,

$$\mathbf{Q}^n = \begin{pmatrix} 2 \cos \varphi & -1 \\ 1 & 0 \end{pmatrix}^n = \frac{1}{\sin \varphi} \begin{pmatrix} \sin([n+1]\varphi) & -\sin(n\varphi) \\ \sin(n\varphi) & -\sin([n-1]\varphi) \end{pmatrix}, \quad (44)$$

and the characteristic polynomial, now in terms of φ , simplifies to

$$P_M(\varphi) = \cos(M\varphi) + \left(t_+ \cos \varphi - t_- \frac{z_+}{z_-} \right) \frac{\sin(M\varphi)}{\sin \varphi} \quad (45)$$

up to an irrelevant factor $2/(t_+ + 1)(t_- z_-/2)^M$. $P_M(\varphi)$ can be written in terms of Chebyshev polynomials of the first and second kind, $T_M(\cos \varphi) = \cos(M\varphi)$ and $U_{M-1}(\cos \varphi) = \sin(M\varphi)/\sin \varphi$, and is therefore a polynomial of degree M in $\cos \varphi$.

Using the characteristic polynomial $P_M(\varphi)$ we can come back to the arrangement of the eigenvalues λ of \mathcal{T}_2 and \mathcal{T} . It turns out that it is beneficial to sort the $2M$ eigenvalues λ of

\mathcal{T}_2 by the value of φ , first selecting the zeroes of $P_M(\varphi)$ with negative slope ordered by $|\varphi|$ (green points in Fig. 2), and then selecting the zeroes of $P_M(\varphi)$ with positive slope ordered by $|\varphi|$ (red points in Fig. 2). Slightly below T_c the two zeroes φ_1 and φ_{M+1} are zero and become complex below [15]. However, the corresponding values λ_1 and λ_{M+1} are always real and define the correct order.

The arrangement is compatible with Eq. (32) and leads to the following identities: From Eq. (43), we derive the identities

$$\sin \frac{\varphi}{2} = -\frac{\sqrt{z-t\lambda}\sqrt{t-z\lambda}}{2\sqrt{tz\lambda}\sqrt{t_-z_-}}, \quad (46a)$$

$$\cos \frac{\varphi}{2} = \frac{\sqrt{\lambda-tz}\sqrt{1-tz\lambda}}{2\sqrt{tz\lambda}\sqrt{t_-z_-}}, \quad (46b)$$

$$\tan \frac{\varphi}{2} = -\frac{\sqrt{z-t\lambda}\sqrt{t-z\lambda}}{\sqrt{\lambda-tz}\sqrt{1-tz\lambda}} \quad (46c)$$

and, using the characteristic polynomial (45),

$$\sin \frac{M\varphi}{2} = \pm \frac{\sqrt{z-t\lambda}\sqrt{1-tz\lambda}}{2\sqrt{tz\lambda}\sqrt{t_-z_-}}, \quad (47a)$$

$$\cos \frac{M\varphi}{2} = \pm \frac{\sqrt{t-z\lambda}\sqrt{\lambda-tz}}{2\sqrt{tz\lambda}\sqrt{t_-z_-}}, \quad (47b)$$

$$\tan \frac{M\varphi}{2} = \frac{\sqrt{z-t\lambda}\sqrt{1-tz\lambda}}{\sqrt{t-z\lambda}\sqrt{\lambda-tz}} \quad (47c)$$

as well as

$$\frac{\sin(M\varphi)}{\sin \varphi} = -\frac{z_-}{\lambda_-}. \quad (48)$$

These identities will be used in the following to simplify the eigenvectors of \mathcal{T} .

VII. EIGENVECTORS OF \mathcal{T}

The common eigenvectors of \mathcal{T} , \mathcal{T}_+ and \mathcal{T}_- can be calculated from the recursion matrix Eq. (44), too, and read

$$\begin{aligned} (\mathbf{x})_{\lambda,n} &= (\vec{x}_\lambda)_n \propto \langle 1, 0 | \mathbf{Q}^n | 1, t^*/z^* \rangle \\ &\propto \frac{\sin([n+1]\varphi)}{(1-t)(1+z)} - \frac{\sin(n\varphi)}{(1+t)(1-z)}, \end{aligned} \quad (49)$$

with $n = 0, \dots, M-1$. After proper normalization and an index change from n to $m = -M+1, -M+3, \dots, M-1$, running over the odd integers between $-M$ and M , the matrix

elements of \mathbf{x} are

$$(\mathbf{x})_{\lambda,m} = \frac{\sqrt{4tz} t_- z_- \lambda_-}{\sqrt{M\lambda_-^2 + z_+ \lambda_+ - t_+} \sqrt{\lambda_+ - 1}} \left[\frac{\sin([M+1+m]\frac{\varphi}{2})}{(1-t)(1+z)} - \frac{\sin([M-1+m]\frac{\varphi}{2})}{(1+t)(1-z)} \right]. \quad (50)$$

The block-diagonal transfer matrix, Eq. (36), enables us to reduce the problem of calculating the partition function from $2M \times 2M$ matrices to $M \times M$ matrices, and to factorize the involved determinants. This will be demonstrated in the following chapter.

VIII. PARTITION FUNCTION FACTORIZATION

Using the eigensystem defined above and the block diagonal form Eq. (36), we can write the partition function Eq. (25b) as

$$\mathcal{Z} = \det \langle \mathbf{S}^+ \mathbf{S}^- | \begin{bmatrix} \mathcal{T}^L & 0 \\ 0 & \mathcal{T}^{-L} \end{bmatrix} | \mathbf{S}^+ \mathbf{S}^- \rangle \quad (51a)$$

$$= \det (\mathbf{S}^+ \mathcal{T}^L \mathbf{S}^+ + \mathbf{S}^- \mathcal{T}^{-L} \mathbf{S}^-), \quad (51b)$$

with $\mathbf{S}^\pm \equiv \frac{1}{2}(\mathbf{1} \pm \mathbf{S})$. At this point we define the $M \times M$ matrix

$$\mathbf{M} \equiv \mathbf{x}(\mathcal{T}^{L/2} \mathbf{S}^+ + \mathcal{T}^{-L/2} \mathbf{S}^-), \quad (52)$$

which completes the square in Eq. (51b), as

$$\begin{aligned} \bar{\mathbf{M}}\mathbf{M} &= (\mathbf{S}^+ \mathcal{T}^{L/2} + \mathbf{S}^- \mathcal{T}^{-L/2}) \bar{\mathbf{x}} \mathbf{x} (\mathcal{T}^{L/2} \mathbf{S}^+ + \mathcal{T}^{-L/2} \mathbf{S}^-) \\ &= \mathbf{S}^+ \mathcal{T}^L \mathbf{S}^+ + \mathbf{S}^+ \mathbf{S}^- + \mathbf{S}^- \mathbf{S}^+ + \mathbf{S}^- \mathcal{T}^{-L} \mathbf{S}^- \\ &= \mathbf{S}^+ \mathcal{T}^L \mathbf{S}^+ + \mathbf{S}^- \mathcal{T}^{-L} \mathbf{S}^- \end{aligned} \quad (53)$$

and the mixed terms in the expansion vanish, $\mathbf{S}^+ \mathbf{S}^- = \mathbf{S}^- \mathbf{S}^+ = \frac{1}{4}(\mathbf{1} - \mathbf{S}^2) = \mathbf{0}$. With $\mathbf{x} \mathcal{T}^{\pm L/2} = \Lambda^{\pm L/2} \mathbf{x}$ from Eq. (34) the matrix elements of \mathbf{M} are

$$(\mathbf{M})_{\lambda,m} = \frac{1}{2}(\lambda^{L/2} + \lambda^{-L/2})(\mathbf{x})_{\lambda,m} + \frac{1}{2}(\lambda^{L/2} - \lambda^{-L/2})(\mathbf{x})_{\lambda,-m}, \quad (54)$$

and the partition function Eq. (51) becomes

$$\mathcal{Z} = \det (\bar{\mathbf{M}}\mathbf{M}) = \det^2 \mathbf{M}, \quad (55)$$

i. e., $Z \propto \det \mathbf{M}$.

We now insert the definition of \mathbf{x} from Eq. (50) and pull out common m -independent factors, primarily the normalization constants, which we can move into a diagonal matrix \mathbf{D} according to

$$\bar{\mathbf{M}}\mathbf{M} \equiv \bar{\mathbf{W}}\mathbf{D}\mathbf{W}. \quad (56)$$

We first choose the decomposition

$$(\mathbf{W}^\dagger)_{\lambda,m} \equiv \frac{1}{2} \sum_{\pm} (\lambda^{L/2} \pm \lambda^{-L/2}) \left(\frac{\sin([M+1 \pm m]\frac{\varphi}{2})}{(1-t)(1+z)} - \frac{\sin([M-1 \pm m]\frac{\varphi}{2})}{(1+t)(1-z)} \right), \quad (57a)$$

$$(\mathbf{D}^\dagger)_{\lambda,\lambda} \equiv \frac{8tz\lambda(t_-z_-\lambda_-)^2}{(M\lambda_-^2 + z_+\lambda_+ - t_+)(1-\lambda)^2}, \quad (57b)$$

and sort $(\mathbf{W}^\dagger)_{\lambda,m}$ by terms in $\lambda^{\pm L/2}$ to get, after some trigonometry,

$$\begin{aligned} (\mathbf{W}^\dagger)_{\lambda,m} = \frac{\sin \varphi}{4tt_-zz_-} & \left[\lambda^{L/2} \left((t-z) \frac{\sin \frac{M\varphi}{2}}{\sin \frac{\varphi}{2}} - (tz-1) \frac{\cos \frac{M\varphi}{2}}{\cos \frac{\varphi}{2}} \right) \cos \frac{m\varphi}{2} \right. \\ & \left. + \lambda^{-L/2} \left((t-z) \frac{\cos \frac{M\varphi}{2}}{\sin \frac{\varphi}{2}} + (tz-1) \frac{\sin \frac{M\varphi}{2}}{\cos \frac{\varphi}{2}} \right) \sin \frac{m\varphi}{2} \right]. \end{aligned} \quad (58)$$

Pulling out some factors and rearranging terms we get

$$\begin{aligned} (\mathbf{W}^\dagger)_{\lambda,m} = \frac{\sin \varphi \cos \frac{M\varphi}{2}}{4tt_-zz_-} & \left[\lambda^{L/2} \left((t-z) \frac{\tan \frac{M\varphi}{2}}{\tan \frac{\varphi}{2}} - (tz-1) \right) \frac{\cos \frac{m\varphi}{2}}{\cos \frac{\varphi}{2}} + \right. \\ & \left. + \lambda^{-L/2} \left((t-z) + (tz-1) \frac{\tan \frac{M\varphi}{2}}{\cot \frac{\varphi}{2}} \right) \frac{\sin \frac{m\varphi}{2}}{\sin \frac{\varphi}{2}} \right]. \end{aligned} \quad (59)$$

Further simplifications occur if we use the identities from Eqs. (46) and (47), especially

$$\frac{\tan \frac{M\varphi}{2}}{\cot \frac{\varphi}{2}} = \frac{z-t\lambda}{tz-\lambda}, \quad \frac{\tan \frac{M\varphi}{2}}{\tan \frac{\varphi}{2}} = \frac{tz\lambda-1}{t-z\lambda}. \quad (60)$$

Shifting again m -independent factors from \mathbf{W}^\dagger to \mathbf{D}^\dagger , the result can be simplified to

$$(\mathbf{W}^\dagger)_{\lambda,m} \equiv \frac{1}{\sqrt{t_-z_-}} \left[\lambda^{L/2} (tz-\lambda) \frac{\cos \frac{m\varphi}{2}}{\cos \frac{\varphi}{2}} - \lambda^{-L/2} (tz^{-1}-\lambda) \frac{\sin \frac{m\varphi}{2}}{\sin \frac{\varphi}{2}} \right] \quad (61a)$$

$$(\mathbf{D})_{\lambda,\lambda} \equiv \frac{\lambda_-}{2z_-} \frac{(t_+z_+ - \lambda_+)^2 - t_-^2z_-^2}{M\lambda_-^2 + z_+\lambda_+ - t_+} \frac{1}{(tz-\lambda)(tz^{-1}-\lambda)}, \quad (61b)$$

and Eq. (55) becomes

$$\mathcal{Z} = \det^2 \mathbf{W}^\dagger \prod_{\lambda} (\mathbf{D})_{\lambda\lambda}. \quad (62)$$

The remaining challenge is the calculation of $\det \mathbf{W}^\dagger$, which will be further simplified in the following.

IX. THE VANDERMONDE DETERMINANT

We now utilize the observation that the matrix \mathbf{W}^\dagger is a Vandermonde matrix, and that its determinant is invariant under basis transformations between complete polynomial bases. Hence we can transform \mathbf{W}^\dagger from the trigonometric basis to the simpler power basis. We identify the leading term in both $\cos \frac{m\varphi}{2} / \cos \frac{\varphi}{2}$ and $\sin \frac{m\varphi}{2} / \sin \frac{\varphi}{2}$ to be¹

$$\frac{\cos \frac{m\varphi}{2}}{\cos \frac{\varphi}{2}} \simeq \left(2 \cos \frac{\varphi}{2}\right)^{|m|-1}, \quad \frac{\sin \frac{m\varphi}{2}}{\sin \frac{\varphi}{2}} \simeq \frac{m}{|m|} \left(2 \cos \frac{\varphi}{2}\right)^{|m|-1} \quad (63)$$

and rewrite the result using Eq. (46b), as $2n \equiv |m| - 1$ is an even integer, to

$$\left(2 \cos \frac{\varphi}{2}\right)^{2n} = \left[\frac{(\lambda - tz)(1 - tz\lambda)}{tz\lambda t_- z_-}\right]^n \simeq \left(\frac{-2}{t_- z_-}\right)^n \lambda_+^n. \quad (64)$$

The determinant becomes

$$\det \mathbf{W}^\dagger = \left(\frac{2}{t_- z_-}\right)^{M^2/2} \det \mathbf{W} \quad (65)$$

with

$$\mathbf{W} = \begin{pmatrix} g_1 c_1^{M/2-1} & \cdots & g_1 c_1 & g_1 & f_1 & f_1 c_1 & \cdots & f_1 c_1^{M/2-1} \\ g_2 c_2^{M/2-1} & \cdots & g_2 c_2 & g_2 & f_2 & f_2 c_2 & \cdots & f_2 c_2^{M/2-1} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ g_M c_M^{M/2-1} & \cdots & g_M c_M & g_M & f_M & f_M c_M & \cdots & f_M c_M^{M/2-1} \end{pmatrix}, \quad (66)$$

where we introduced the abbreviations

$$c_\mu \equiv \lambda_{\mu,+}, \quad g_\mu \equiv -\lambda_\mu^{L/2}(tz - \lambda_\mu), \quad f_\mu \equiv \lambda_\mu^{-L/2}(tz^{-1} - \lambda_\mu). \quad (67)$$

Using a block Laplace expansion along the vertical line in Eq. (66), the determinant of \mathbf{W} can be written as alternating sum over all possible $M/2 \times M/2$ g -minors $\det \mathbf{W}_{\mathbf{s}, \{1, \dots, M/2\}}$, times the corresponding f -minors $\det \mathbf{W}_{\bar{\mathbf{s}}, \{M/2+1, \dots, M\}}$,

$$\det \mathbf{W} = \pm \sum_{\mathbf{s}} \text{sign}(\mathbf{s}, \bar{\mathbf{s}}) \underbrace{\prod_{\mu \in \mathbf{s}} g_\mu \prod_{\mu < \nu \in \mathbf{s}} (c_\mu - c_\nu)}_{\det \mathbf{W}_{\mathbf{s}, \{1, \dots, M/2\}}} \underbrace{\prod_{\mu \in \bar{\mathbf{s}}} f_\mu \prod_{\mu < \nu \in \bar{\mathbf{s}}} (c_\mu - c_\nu)}_{\det \mathbf{W}_{\bar{\mathbf{s}}, \{M/2+1, \dots, M\}}} \quad (68)$$

where \mathbf{s} denotes one of the $\binom{M}{M/2}$ possible subsets of $M/2$ choices of the index set $\{1, \dots, M\}$, and $\bar{\mathbf{s}}$ its complement. Both minors are simple Vandermonde determinants, and the irrelevant overall sign depends on the ordering within the sets.

¹ “ \simeq ” denoted “asymptotically equal”

In the following, we further reduce the matrix size from $M \times M$ to $M/2 \times M/2$ by Vandermonde-type row elimination. While for simple Vandermonde determinants this procedure leads a complete factorization, in our case we can only eliminate $M/2$ rows, which we nevertheless can choose arbitrary. We now denote the chosen set of eliminated rows and its complement by \mathbf{s} and $\bar{\mathbf{s}}$ and find ($\mathbf{A}_{\mathbf{s}} \equiv \mathbf{A}_{\mathbf{s},\mathbf{s}}$)

$$\det \mathbf{W} = \pm d_{\mathbf{s},\bar{\mathbf{s}}} \det (\mathbf{G}_{\mathbf{s}} \mathbf{T}_{\mathbf{s},\bar{\mathbf{s}}} \mathbf{F}_{\bar{\mathbf{s}}} - \mathbf{F}_{\mathbf{s}} \mathbf{T}_{\mathbf{s},\bar{\mathbf{s}}} \mathbf{G}_{\bar{\mathbf{s}}}), \quad (69)$$

with the $M \times M$ matrices

$$(\mathbf{G})_{\mu\mu} \equiv g_{\mu}, \quad (\mathbf{F})_{\mu\mu} \equiv f_{\mu}, \quad (\mathbf{T})_{\mu\nu} \equiv \frac{1}{c_{\mu} - c_{\nu}}, \quad (70a)$$

(we can set $(\mathbf{T})_{\mu\mu} \equiv 0$, as $\mu \neq \nu$), and with the double product

$$d_{\mathbf{s},\bar{\mathbf{s}}} \equiv \prod_{\mu \in \mathbf{s}} \prod_{\nu \in \bar{\mathbf{s}}} (c_{\mu} - c_{\nu}). \quad (70b)$$

As $\mathbf{T}_{\mathbf{s},\bar{\mathbf{s}}}$ is a Cauchy matrix, both $\mathbf{G}_{\mathbf{s}} \mathbf{T}_{\mathbf{s},\bar{\mathbf{s}}} \mathbf{F}_{\bar{\mathbf{s}}}$ and $\mathbf{F}_{\mathbf{s}} \mathbf{T}_{\mathbf{s},\bar{\mathbf{s}}} \mathbf{G}_{\bar{\mathbf{s}}}$ are Cauchy-like matrices. An example with $M = 6$ and $\mathbf{s} = \{1, 3, 5\}$, such that $\bar{\mathbf{s}} = \{2, 4, 6\}$, reads

$$\mathbf{G}_{\mathbf{s}} = \begin{pmatrix} g_1 & & \\ & g_3 & \\ & & g_5 \end{pmatrix}, \quad \mathbf{F}_{\bar{\mathbf{s}}} = \begin{pmatrix} f_2 & & \\ & f_4 & \\ & & f_6 \end{pmatrix}, \quad \mathbf{T}_{\mathbf{s},\bar{\mathbf{s}}} = \begin{pmatrix} \frac{1}{c_1 - c_2} & \frac{1}{c_3 - c_2} & \frac{1}{c_5 - c_2} \\ \frac{1}{c_1 - c_4} & \frac{1}{c_3 - c_4} & \frac{1}{c_5 - c_4} \\ \frac{1}{c_1 - c_6} & \frac{1}{c_3 - c_6} & \frac{1}{c_5 - c_6} \end{pmatrix}. \quad (71)$$

The choice of \mathbf{s} has influence on the magnitude of the two terms in Eq. (69) and has a physical interpretation: If we choose $\mathbf{s} = \mathbf{o} \equiv \{1, 3, \dots, M-1\}$ odd integers, both $\mathbf{G}_{\mathbf{s}}$ and $\mathbf{F}_{\bar{\mathbf{s}}}$ contain only dominant (for large L) eigenvalues $\lambda_{\mu} > 1$, while the subdominant ones $\lambda_{\mu} < 1$ enter $\mathbf{G}_{\bar{\mathbf{s}}}$ and $\mathbf{F}_{\mathbf{s}}$. Therefore, the term $\mathbf{G}_{\mathbf{s}} \mathbf{T}_{\mathbf{s},\bar{\mathbf{s}}} \mathbf{F}_{\bar{\mathbf{s}}}$ in Eq. (69) gives the leading contribution for large L , and the second one $\mathbf{F}_{\mathbf{s}} \mathbf{T}_{\mathbf{s},\bar{\mathbf{s}}} \mathbf{G}_{\bar{\mathbf{s}}}$ the finite- L corrections. The oscillating behavior

$$\text{sign} \log \lambda_{\mu} = \text{sign} \gamma_{\mu} = \text{sign} \varphi_{\mu} = (-1)^{\mu-1}, \quad \mu = 1, \dots, M, \quad (72)$$

is dictated by the ordering of the zeroes of $P_M(\varphi)$, Eq. (45), as described above.

Consequently, we factor out the leading first term of the determinant Eq. (69),

$$\det \mathbf{W} = \pm d_{\mathbf{s},\bar{\mathbf{s}}} \det (\mathbf{G}_{\mathbf{s}} \mathbf{T}_{\mathbf{s},\bar{\mathbf{s}}} \mathbf{F}_{\bar{\mathbf{s}}}) \det (\mathbf{1} - \mathbf{F}_{\bar{\mathbf{s}}}^{-1} \mathbf{T}_{\mathbf{s},\bar{\mathbf{s}}}^{-1} \mathbf{G}_{\mathbf{s}}^{-1} \mathbf{F}_{\mathbf{s}} \mathbf{T}_{\mathbf{s},\bar{\mathbf{s}}} \mathbf{G}_{\bar{\mathbf{s}}}), \quad (73)$$

and express the inverse $\mathbf{T}_{\mathbf{s},\bar{\mathbf{s}}}^{-1}$ through the diagonal matrix

$$(\mathbf{P})_{\mu\mu} \equiv p_{\mu} \equiv \prod_{\nu=1}^M (c_{\mu} - c_{\nu})^{-\sigma_{\mu}\sigma_{\nu}}, \quad (74)$$

which fulfills

$$\mathbf{P}_{\bar{s}} \mathbf{T}_{\bar{s},s} \mathbf{P}_s \mathbf{T}_{s,\bar{s}} = \mathbf{1}. \quad (75)$$

Here, \prod' denotes the regularized product, with zero and infinite factors removed, and we have defined the parity of μ

$$\sigma_\mu \equiv \begin{cases} +1 & \text{if } \mu \in \mathbf{s} \\ -1 & \text{if } \mu \in \bar{\mathbf{s}}. \end{cases} \quad (76)$$

We now introduce the diagonal matrix

$$(\mathbf{V})_{\mu\mu} \equiv v_\mu \equiv -p_\mu \lambda_\mu^L \left(\frac{g_\mu}{f_\mu} \right)^{\sigma_\mu} = p_\mu \frac{tz^{-\sigma_\mu} - \lambda_\mu}{tz^{\sigma_\mu} - \lambda_\mu} \quad (77)$$

and define, with $\mathbf{\Lambda}$ from Eq. (33), for the specific set of dominant odd indices \mathbf{o} as well as the complementary set of even indices $\mathbf{e} \equiv \bar{\mathbf{o}}$ the residual matrix

$$\mathbf{Y} \equiv -\mathbf{\Lambda}_{\mathbf{e}}^L \mathbf{V}_{\mathbf{e}} \mathbf{T}_{\mathbf{e},\mathbf{o}} \mathbf{\Lambda}_{\mathbf{o}}^{-L} \mathbf{V}_{\mathbf{o}} \mathbf{T}_{\mathbf{o},\mathbf{e}} \quad (78)$$

to find

$$\det \mathbf{W} = \pm d_{\mathbf{o},\mathbf{e}} \det \mathbf{T}_{\mathbf{o},\mathbf{e}} \det \mathbf{G}_{\mathbf{o}} \det \mathbf{F}_{\mathbf{e}} \det(\mathbf{1} + \mathbf{Y}). \quad (79)$$

Remember that the matrices with one index are diagonal. The determinant of the Cauchy matrix $\mathbf{T}_{\mathbf{s},\bar{\mathbf{s}}}$ reads

$$\det \mathbf{T}_{\mathbf{s},\bar{\mathbf{s}}} = \pm \frac{q_{\mathbf{s}} q_{\bar{\mathbf{s}}}}{d_{\mathbf{s},\bar{\mathbf{s}}}}, \quad (80)$$

with

$$q_{\mathbf{s}} \equiv \prod_{\mu < \nu \in \mathbf{s}} (c_\mu - c_\nu), \quad (81)$$

leading to the final form

$$\det \mathbf{W} = |q_{\mathbf{o}} q_{\mathbf{e}}| \det \mathbf{G}_{\mathbf{o}} \det \mathbf{F}_{\mathbf{e}} \det(\mathbf{1} + \mathbf{Y}). \quad (82)$$

X. RESULTING PARTITION FUNCTION

Introducing the strip residual partition function

$$Z_{\text{strip}}^{\text{res}} \equiv \det(\mathbf{1} + \mathbf{Y}) \quad (83)$$

for the remaining determinant, and inserting explicit values for $\det \mathbf{G}_{\mathbf{o}}$, $\det \mathbf{F}_{\mathbf{e}}$ and $\det \mathbf{D}$, we arrive at the final result

$$Z = \left[C_3 d_{\mathbf{o},\mathbf{e}}^2 \prod_{\mu=1}^M \frac{(t_+ z_+ - \lambda_{\mu,+})^2 - t_-^2 z_-^2}{M \lambda_{\mu,-}^2 + z_+ \lambda_{\mu,+} - t_+} \frac{\sigma_\mu \lambda_{\mu,-}}{v_\mu} \lambda_\mu^{\sigma_\mu L} \right]^{1/2} Z_{\text{strip}}^{\text{res}}. \quad (84a)$$

for the partition function, with parity $\sigma_\mu = (-1)^{\mu-1}$, $d_{\mathbf{o},\mathbf{e}}$ from Eq. (70b), and the constant

$$C_3 \equiv z^M \left(\frac{2}{z_-} \right)^{LM} \left(\frac{2}{t_- z_-} \right)^{M^2/2}. \quad (84b)$$

We can discuss two limiting cases with respect to the aspect ratio ρ : By definition, the matrix \mathbf{Y} only contains subdominant finite- L contributions, and therefore $\lim_{\rho \rightarrow \infty} \mathbf{Y} = \mathbf{0}$ and $\lim_{\rho \rightarrow \infty} Z_{\text{strip}}^{\text{res}} = 1$. On the other hand, the limit $\rho \rightarrow 0$ can also be discussed. As L is a real number in Eq. (78), we can let $L \rightarrow 0$ and find a Cauchy-type matrix very similar to one describing the spontaneous magnetization of the superintegrable chiral Potts model, as discussed by Baxter [21]. The resulting determinant can be factorized and reads

$$\lim_{\rho \rightarrow 0} Z_{\text{strip}}^{\text{res}} = \pm \frac{(2tz_-)^{M/2} \prod_{\mu \in \mathbf{o}} \prod_{\nu \in \mathbf{e}} (\lambda_\mu - \lambda_\nu)}{M \prod_{\mu=1} (tz^{\sigma_\mu} - \lambda_\mu) \prod_{\mu < \nu \in \mathbf{o}} (1 - \lambda_\mu \lambda_\nu) \prod_{\mu < \nu \in \mathbf{e}} (1 - \lambda_\mu \lambda_\nu)}. \quad (85)$$

To summarize, we find closed product representations for both limit cases $L/M \rightarrow \infty$ and $M/L \rightarrow \infty$ with finite M . The general case $0 < \rho < \infty$, however, involves a nontrivial determinant, Eq. (83).

The oscillating order of the eigenvalues introduced in Chapter V was a prerequisite for the simple block diagonalization of the block transfer matrix \mathcal{T}_2 , Eq. (36), and the subsequent factorization of Z . However, now we observe that this oscillation is reversed by the sets \mathbf{o} and \mathbf{e} of odd and even indices, used in the definition of the residual matrix \mathbf{Y} . Therefore, we rewrite the results Eqs. (78) and (84a) in terms of the simpler non-oscillating dominant eigenvalues $\hat{\lambda}_\mu$. Using the parity $\sigma_\mu = (-1)^{\mu-1}$, we define²

$$\hat{\lambda}_\mu \equiv \lambda_\mu^{\sigma_\mu} > 1, \quad \hat{\gamma}_\mu \equiv \sigma_\mu \gamma_\mu = |\gamma_\mu| > 0, \quad \hat{\varphi}_\mu \equiv \sigma_\mu \varphi_\mu \stackrel{\mu \geq 1}{=} |\varphi_\mu| > 0, \quad \mu = 1, \dots, M, \quad (86)$$

implying $\hat{\lambda}_{\mu,+} = \lambda_{\mu,+} = \hat{c}_\mu = c_\mu$ and $\hat{\lambda}_{\mu,-} = \sigma_\mu \lambda_{\mu,-} = |\lambda_{\mu,-}|$, to get

$$\hat{v}_\mu = v_\mu = p_\mu \frac{tz^{-\sigma_\mu} - \hat{\lambda}_\mu^{\sigma_\mu}}{tz^{\sigma_\mu} - \hat{\lambda}_\mu^{\sigma_\mu}}, \quad (87a)$$

$$\hat{\mathbf{Y}} = \mathbf{Y} = -\hat{\mathbf{A}}_{\mathbf{e}}^{-L} \mathbf{V}_{\mathbf{e}} \mathbf{T}_{\mathbf{e},\mathbf{o}} \hat{\mathbf{A}}_{\mathbf{o}}^{-L} \mathbf{V}_{\mathbf{o}} \mathbf{T}_{\mathbf{o},\mathbf{e}}, \quad (87b)$$

leading to the partition function in terms of $\hat{\lambda}_\mu$,

$$Z = \left[C_3 d_{\mathbf{o},\mathbf{e}}^2 \prod_{\mu=1}^M \frac{(t_+ z_+ - \hat{\lambda}_{\mu,+})^2 - t_-^2 z_-^2}{M \hat{\lambda}_{\mu,-}^2 + z_+ \hat{\lambda}_{\mu,+} - t_+} \frac{\hat{\lambda}_{\mu,-}}{v_\mu} \hat{\lambda}_\mu^L \right]^{1/2} \det(\mathbf{1} + \mathbf{Y}). \quad (87c)$$

² remember that φ_1 becomes imaginary below T_c

This is the final result of our analysis for arbitrary temperature T and finite system size L and M . We factorized the partition function up to the factor $Z_{\text{strip}}^{\text{res}}$, Eq. (83), where the residual matrix \mathbf{Y} contains all information about the finite aspect ratio ρ and will be analyzed in detail in [15]. The first term in Eq. (87c) is the infinite strip contribution, which has been analyzed in great detail by R. J. Baxter recently [17].

XI. FREE ENERGY CONTRIBUTIONS

In this chapter we give a decomposition of the reduced free energy (in units of $k_{\text{B}}T$)

$$F(T; L, M) = -\log Z \quad (88)$$

appropriate for our geometry and method. We first recall that

$$F(T; L, M) = F_{\infty}(T; L, M) + F_{\infty}^{\text{res}}(T; L, M), \quad (89)$$

with infinite volume contribution F_{∞} , that for our geometry has the form

$$F_{\infty}(T; L, M) \equiv LMf_{\text{b}}(T) + Lf_{\text{s}}^{\leftrightarrow}(T) + Mf_{\text{s}}^{\updownarrow}(T) + f_{\text{c}}(T), \quad (90)$$

and can be viewed as a regularization term in the limit $L, M \rightarrow \infty$. The bulk free energy per spin $f_{\text{b}}(T)$, surface free energies per surface spin pair $f_{\text{s}}^{\delta}(T)$, and corner free energy $f_{\text{c}}(T)$ are defined in the thermodynamic limit $L, M \rightarrow \infty$ and do not depend on L, M . However, the residual free energy F_{∞}^{res} , denoted $O(e^{-\gamma L}, e^{-\gamma M})$ in Eq. (1.1) of [17], gives rise to important finite-size effects, most prominently the Casimir amplitude and the critical Casimir force [15].

In the limit $L \rightarrow \infty$ with fixed M , the strip residual partition function $Z_{\text{strip}}^{\text{res}} \rightarrow 1$, as shown in the last chapter. Consequently, we denote the infinite strip contribution

$$Z_{\text{strip}} \equiv Z/Z_{\text{strip}}^{\text{res}} \quad (91)$$

and get a free energy decomposition slightly different from Eq. (89), namely

$$F(T; L, M) = F_{\text{strip}}(T; L, M) + F_{\text{strip}}^{\text{res}}(T; L, M), \quad (92)$$

where we can identify the strip residual free energy

$$F_{\text{strip}}^{\text{res}} \equiv -\log \det(\mathbf{1} + \mathbf{Y}) \quad (93)$$

as the L -dependent term in the difference between the residual free energy F_∞^{res} of the finite rectangular system and the leading divergent term $\mathcal{O}(L)$ in the limit $L \rightarrow \infty$ [13],

$$F_{\text{strip}}^{\text{res}}(T; L, M) = F_\infty^{\text{res}}(T; L, M) - L \lim_{L \rightarrow \infty} L^{-1} F_\infty^{\text{res}}(T; L, M) - F_{\text{s,c}}^{\text{res}}(T; M). \quad (94)$$

Note that the last term $F_{\text{s,c}}^{\text{res}}(T; M)$ drops out in the L -derivative below, for details we refer to [15]. In this notation, Vernier & Jacobsen [16] conjectured a product representation of the infinite volume contribution $Z_\infty \equiv e^{-F_\infty}$ that trivially depends on the system size L and M , see also Appendix A, while Baxter derived a product formula for the infinite strip contribution Z_{strip} at finite strip width M , and then performed the limit $M \rightarrow \infty$ [17]. Both results applied only to the ordered phase below T_c .

Finally we turn to the critical Casimir force. The reduced Casimir force per area M in L direction reads

$$\mathcal{F}(T; L, M) \equiv -\frac{1}{M} \frac{\partial}{\partial L} F_\infty^{\text{res}}(T; L, M) \quad (95)$$

and can be decomposed into two parts to find, in analogy to Eq. (94), the differential contribution

$$\mathcal{F}_{\text{strip}}(T; L, M) \equiv -\frac{1}{M} \frac{\partial}{\partial L} F_{\text{strip}}^{\text{res}}(T; L, M) \quad (96a)$$

$$= \mathcal{F}(T; L, M) + \frac{1}{M} \lim_{L \rightarrow \infty} L^{-1} F_\infty^{\text{res}}(T; L, M). \quad (96b)$$

This contribution is therefore directly related to the remaining determinant Eq. (83). For details on the involved universal amplitudes and finite-size scaling functions the reader again is referred to [15].

XII. EFFECTIVE SPIN MODEL

In this last chapter we present an exact mapping of the residual determinant $Z_{\text{strip}}^{\text{res}}$, Eq. (83), onto an effective spin model with M spins and long-range pair interactions. This model might be a starting point for further investigations of the residual determinant. The mapping is motivated by the observation that the determinant expansion of Eq. (83) is of the form (here we set $L = 0$ for simplicity)

$$Z_{\text{strip}}^{\text{res}} = 1 + \underbrace{\sum_{\mu \in \mathbf{o}} \sum_{\nu \in \mathbf{e}} \frac{v_\mu v_\nu}{(c_\mu - c_\nu)^2}}_{1^{\text{st}} \text{ order}} + \underbrace{\sum_{\mu \neq \mu' \in \mathbf{o}} \sum_{\nu \neq \nu' \in \mathbf{e}} \frac{v_\mu v_{\mu'} v_\nu v_{\nu'} (c_\mu - c_{\mu'})^2 (c_\nu - c_{\nu'})^2}{(c_\mu - c_\nu)^2 (c_\mu - c_{\nu'})^2 (c_{\mu'} - c_\nu)^2 (c_{\mu'} - c_{\nu'})^2}}_{2^{\text{nd}} \text{ order}} + \dots \quad (97)$$

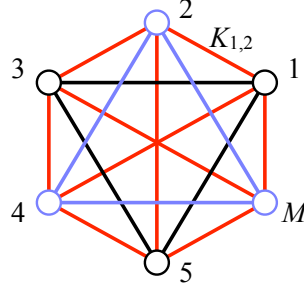


Figure 3. Effective spin model for $M = 6$. The two sublattices of odd and even spins are shown as black and light blue circles. The black and light blue interactions between spins of same parity are ferromagnetic, while the red couplings are antiferromagnetic. Note that the spatial arrangement of the spins is arbitrary, as all couplings $K_{\mu\nu}$ are different.

and consists of $\binom{M}{M/2}$ positive terms. Hence we identify these terms with the Boltzmann factors $e^{-\mathcal{H}_{\text{eff}}}$ of the $\binom{M}{M/2}$ possible spin configurations of M spins $s_\mu \in \{0, 1\}$ under the constraint

$$\sum_{\mu \in \mathbf{o}} s_\mu = \sum_{\nu \in \mathbf{e}} s_\nu \quad \Leftrightarrow \quad \sum_{\mu=1}^M \sigma_\mu s_\mu = 0. \quad (98)$$

We interpret \mathbf{o} and \mathbf{e} as two sublattices, discriminated by the parity σ_μ , Eq. (76), see Fig. 3.

The effective spin model then has the Hamiltonian

$$\mathcal{H}_{\text{eff}} = - \sum_{\mu < \nu=1}^M K_{\mu\nu} s_\mu s_\nu + L \sum_{\mu=1}^M \hat{\gamma}_\mu s_\mu + b \left[\sum_{\mu=1}^M \sigma_\mu s_\mu \right]^2, \quad (99)$$

with interaction constants

$$K_{\mu\nu} = -\sigma_\mu \sigma_\nu \log \frac{v_\mu v_\nu}{(c_\mu - c_\nu)^2}, \quad (100)$$

while the positive $\hat{\gamma}_\mu$ from Eq. (86) play the role of magnetic moments in a homogeneous magnetic field of strength $-L$. Both the couplings $K_{\mu\nu}$ as well as the magnetic moments $\hat{\gamma}_\mu$ depend on the temperature of the underlying Ising model, and the limit $b \rightarrow \infty$ enforces the constraint (98). As $(c_\mu - c_\nu)^2 > v_\mu v_\nu$ for all μ, ν , the couplings $K_{\mu\nu}$ are ferromagnetic for spins within the same set and antiferromagnetic between different sets, as shown in Fig. 3. For $L > 0$, the external magnetic field is antiparallel to the spins and favors states with small magnetization. Consequently, for magnetic field $L \rightarrow \infty$ all spins are forced to have $s_\mu = 0$.

With these definitions, the residual determinant, Eq. (83), is equal to the partition function of the Hamiltonian Eq. (99) in the limit $b \rightarrow \infty$,

$$Z_{\text{strip}}^{\text{res}} = Z_{\text{eff}} \equiv \text{Tr} e^{-\mathcal{H}_{\text{eff}}}, \quad (101)$$

where the trace effectively runs over the $\binom{M}{M/2}$ spin states compatible with condition Eq. (98), and Eq. (97) coincides with the expansion of Z_{eff} around the high-field limit $L \rightarrow \infty$. In this expansion we start with $s_\mu = 0$ ($Z_{\text{eff}} = 1$) and flip one spin in both sublattices to get the first order term. For two reversed spins in both subsystems we find the second order term, and so on.

On the other hand, the zero-field case $L = 0$ is described by Eq. (85), which means that we have found a closed form solution for the partition function of Eq. (99) at vanishing applied field.

The Casimir quantities translate into the effective model as follows: The strip Casimir potential, or strip residual free energy, Eq. (93), is simply the free energy of the effective model Eq. (99),

$$F_{\text{strip}}^{\text{res}}(T; L, M) = -\log Z_{\text{strip}}^{\text{res}} = -\log \text{Tr} e^{-\mathcal{H}_{\text{eff}}}. \quad (102)$$

By the definition Eq. (96a), the differential Casimir force per surface area M is given by

$$\begin{aligned} \mathcal{F}_{\text{strip}}(T; L, M) &= -\frac{1}{M} \frac{\partial}{\partial L} F_{\text{strip}}^{\text{res}}(T; L, M) \\ &= \frac{1}{M} \frac{\partial}{\partial L} \log \text{Tr} e^{-\mathcal{H}_{\text{eff}}} \\ &= -\frac{1}{M} \left\langle \sum_{\mu=1}^M \hat{\gamma}_\mu s_\mu \right\rangle_{\text{eff}} \equiv -m_{\text{eff}}(L), \end{aligned} \quad (103)$$

and is therefore identical to minus the field dependent magnetization per spin of the effective model in an antiparallel magnetic field of strength L .

From this mapping, one could conclude that the residual determinant $Z_{\text{strip}}^{\text{res}}$ cannot be factorized into a product for arbitrary L , as this would imply an exact solution of a spin system with long range frustrated interactions in a magnetic field. However, the couplings Eq. (100) are products of symmetric functions of the c_μ , which might be utilized to find a factorization. In the finite-size scaling limit $L, M \rightarrow \infty$, $T \rightarrow T_c$, at fixed temperature scaling variable $x \propto (T/T_c - 1)L$ and aspect ratio ρ , such a factorization indeed exists at least at the critical point T_c . In this limit, the residual determinant, Eq. (83), can be written

in terms of the Dedekind eta function [15], confirming a result from conformal field theory [12].

XIII. CONCLUSIONS

We have calculated the partition function of the two-dimensional anisotropic square lattice Ising model on a $L \times M$ rectangle with open boundary conditions. The final expression, Eqs. (87), involves M eigenvalues $\hat{\lambda}_\mu$ of a $M \times M$ transfer matrix, represented as zeroes of its characteristic polynomial, Eq. (45). The remaining residual part, Eq. (83), is reduced to the determinant of a $M/2 \times M/2$ matrix, for which we could not find a closed product representation (see also [22]).

An analogous calculation, with similar result Eqs. (87), can be done for arbitrary coupling distributions $z_{\ell < L, m} = z_m$, $t_{\ell, m} = t_m$, as long as the involved transfer matrix \mathcal{T}_2 , Eq. (27), is independent of ℓ . The characteristic polynomial, eigenvalues and eigenvectors of \mathcal{T} , Eq. (34), will however be more complicated. On the other hand, we can return to cylinder geometry with periodic or antiperiodic boundary conditions, $z_{\ell < L, m} = z$, $t_{\ell, m < M} = t$, $t_{\ell, M} = t^{\pm 1}$, in which case the characteristic polynomial, Eq. (45), simply becomes $P_M^{(\text{pbc})}(\varphi) = \cos(\frac{M\varphi}{2})$ or $P_M^{(\text{apbc})}(\varphi) = \sin(\frac{M\varphi}{2})$ independent of temperature, which greatly simplifies the calculations.

An intermediate result, Eq. (25), gives the exact partition function of the Ising model with arbitrary couplings $K_{\ell, m}^{\leftrightarrow}$ and $K_{\ell, m}^{\uparrow}$ on the cylinder in terms of a product of very simple 2×2 block transfer matrices with $M \times M$ blocks. This representation can be used to, e.g., investigate diluted systems, or to exactly determine the critical Casimir potential and force between extended particles on the lattice, as introduced in [23, 24]. Due to the reduction to $2M \times 2M$ matrices, numerically exact calculations are possible for large systems up to $M \approx 1000$ and arbitrary L on actual personal computers. However, depending on the actual coupling configuration it might be necessary to use extended numerical precision.

Finally, we presented an exact mapping of the residual part $Z_{\text{strip}}^{\text{res}}$ of the partition function onto an effective spin system with long-range frustrated interactions in an external magnetic field of strength L . This model might serve as starting point for further investigations.

The finite-size scaling limit of the considered model, as well as results for the Casimir potential and Casimir force scaling functions, will be published in the second part of this work [15].

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Appendix A: Product formulas for free energy contributions

In this appendix we will give, without derivation, the product formulas for the singular parts of the free energies f_b , f_s and f_c above and below T_c for the isotropic Ising model, where $K = K^{\leftrightarrow} = K^{\uparrow}$ and $z = t^*$. The calculation is done similar to [16]: Using the finite lattice method [25] we generate the high- and low-temperature series expansion of the free energies up to some finite order and rewrite the series in terms of the natural variable q [26] using the inverse Euler transform [27]. Interestingly, both the finite lattice method and the inverse Euler transform are based on the Möbius inversion formula from elementary number theory [28]. The resulting infinite product in q has a periodic structure

$$\prod_{k=1}^{\infty} (1 - q^k)^{c_{0,k} + c_{1,k}}, \quad (\text{A1})$$

i. e., the coefficients $c_{0,k}$ and $c_{1,k}$ are oscillating sequences, with period $p \in \{4, 8, 16\}$, which can be identified. These sequences are then conjectured to continue to $k \rightarrow \infty$. First we recall the results of Vernier & Jacobsen [16] obtained for temperatures below T_c .

Infinite products like (A1) can be written in many different ways. For the sake of clarity we first introduce a simple notation for such periodic products: We define the function

$$\Pi(\mathbf{C}|q) \equiv \prod_{k=1}^{\infty} (1 - q^k)^{c_k}, \quad (\text{A2})$$

where the $(m+1) \times p$ coefficient matrix \mathbf{C} defines the m^{th} -order polynomials

$$c_k = \sum_{j=0}^m \mathbf{C}_{j, k \bmod p} k^j. \quad (\text{A3})$$

With this definition we first rewrite the results of Vernier & Jacobsen: The natural low

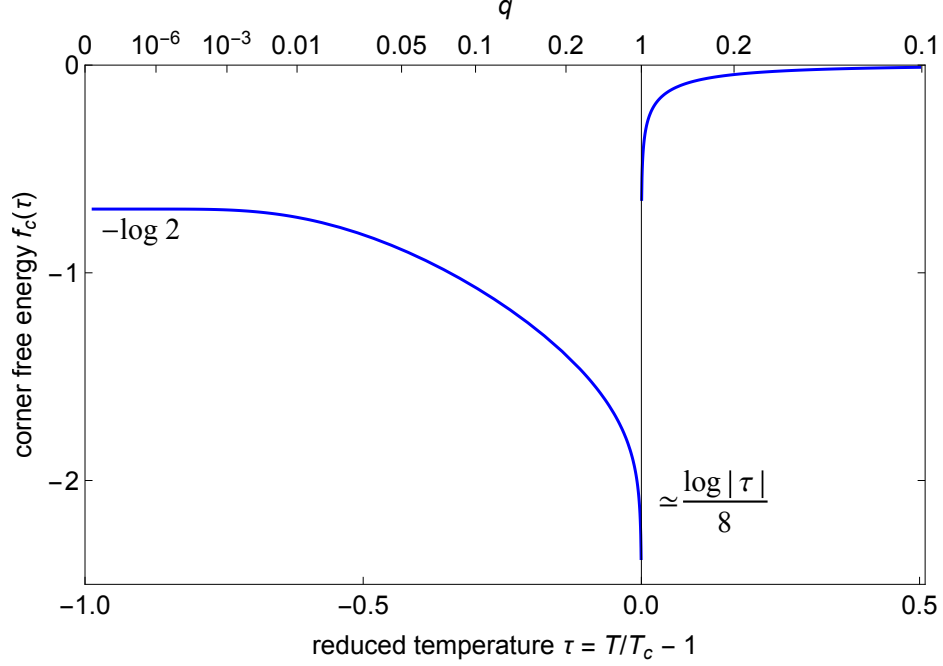


Figure 4. Corner free energy f_c vs. reduced temperature of the two-dimensional Ising model. The corresponding values of the natural variable q are shown at the upper frame.

temperature variable q fulfills [16, Eq. (48)]

$$\begin{aligned}
 t^< &= \sqrt{q} \Pi \left(\begin{matrix} 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \end{matrix} \middle| q \right) \\
 &= \sqrt{q} \frac{(q; q^8)_\infty (q^7; q^8)_\infty}{(q^3; q^8)_\infty (q^5; q^8)_\infty} = \sqrt{q} \frac{(1-q^1)(1-q^7)(1-q^9)(1-q^{15}) \dots}{(1-q^3)(1-q^5)(1-q^{11})(1-q^{13}) \dots},
 \end{aligned} \tag{A4}$$

where $t^< = e^{-2K^<}$, and $(a; q)_\infty$ denotes the q -Pochhammer symbol. Then, the singular bulk, surface and corner free energies become [16, Eq. (49)]

$$e^{-f_{b,\text{sing}}^<} = \frac{1}{\sqrt{q}} \Pi \left(\begin{matrix} 0 & 0 & -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{matrix} \middle| q \right), \tag{A5a}$$

$$e^{-f_{s,\text{sing}}^<} = \frac{1}{2} \Pi \left(\begin{matrix} 0 & \frac{3}{4} & -1 & -\frac{3}{4} & 2 & -\frac{3}{4} & -1 & \frac{3}{4} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} \end{matrix} \middle| q \right) \Pi \left(\begin{matrix} 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{matrix} \middle| \sqrt{q} \right), \tag{A5b}$$

$$e^{-f_c^<} = 2 \Pi \left(\begin{matrix} 0 & -2 & 3 & -2 & -1 & -2 & 3 & -2 \\ 0 & -2 & \frac{1}{2} & 2 & 0 & -2 & -\frac{1}{2} & 2 \end{matrix} \middle| q \right), \tag{A5c}$$

where the regular part of f_c is zero, while $f_{b,\text{reg}} = -\log[2(1+z^2)/(1-z^2)]$ and $f_{s,\text{reg}} = -\frac{1}{4} \log(1-z^2)$. Doing the same analysis in the paramagnetic phase we first identify the high

temperature variable $z^> = \tanh K^>$ by duality [26], such that

$$z^> = \sqrt{q} \Pi \left(0 \ 1 \ 0 \ -1 \ 0 \ -1 \ 0 \ 1 \mid q \right) \quad (\text{A6})$$

has the same product representation as Eq. (A4). Then we find the infinite products

$$e^{-f_{\text{b,sing}}^>} = \Pi \left(\begin{array}{cccccc} 0 & 2 & -4 & 2 & 0 & 2 & -4 & 2 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{array} \mid q \right) = \Pi \left(\begin{array}{cccc} 0 & 2 & -4 & 2 \\ 0 & -1 & 0 & 1 \end{array} \mid q \right), \quad (\text{A7a})$$

$$e^{-f_{\text{s,sing}}^>} = \Pi \left(\begin{array}{cccccc} 0 & \frac{1}{4} & 1 & -\frac{1}{4} & -2 & -\frac{1}{4} & 1 & \frac{1}{4} \\ 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{array} \mid q \right), \quad (\text{A7b})$$

$$e^{-f_c^>} = \Pi \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \end{array} \mid q \right) = \Pi \left(\begin{array}{cccc} 0 & 0 & -3 & 0 \\ 0 & -1 & 0 & 1 \end{array} \mid q^2 \right) \quad (\text{A7c})$$

$$= \prod_{k=0}^{\infty} \frac{1}{(1 - q^{2(4k+2)})^3} \frac{(1 - q^{2(4k+3)})^{4k+3}}{(1 - q^{2(4k+1)})^{4k+1}}. \quad (\text{A7d})$$

Note that the period of all three products above T_c is half of the period below T_c ($e^{-f_{\text{s,sing}}^<}$ can be written as a single product in \sqrt{q} , with period 16). The second product in $e^{-f_{\text{s,sing}}^<}$ is interpreted as the additional contribution from the surface tension. The corner free energy $f_c^>$ can be written as a function of q^2 , because all coefficients c_k are even numbers. Finally, we show the corner free energy f_c in Fig. 4. For $T \rightarrow 0$, $f_c \rightarrow -\log 2$, while for $T \rightarrow T_c$ we find a logarithmic divergence from both sides, with different amplitudes. A detailed discussion of the critical region will be presented in [15].

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